Generalized Correlation for Bubble Motion

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The motion of a single gas bubble rising steadily in an uncontaminated Newtonian viscous fluid of infinite extent is reviewed. Using experimental data gathered from the literature over the past century, a data bank is set up for bubble velocities as a function of their diameters. These data provided a very broad range of liquid properties and bubble hydrodynamic regimes as density (722–1,380 kg/m^3), viscosity (2.2×10^{-4}–18 Pa·s), surface tension (15.9–91.0 mN/m), Reynolds number (1.9×10^{-7}–1.1×10^4), and Morton number (1.0×10^{-11}–1.0×10^7). From a dimensional analysis, a single equation is proposed to predict bubble velocities. The simplicity and good precision over the whole range of parameters facilitate its use in design calculations of chemical engineering.

Introduction

The motion of single gas bubbles in a viscous liquid medium has been the subject of many studies in the past due to its direct implication in the design of gas–liquid contactors. Examples of applications are in wastewater treatments, fermentation processes, and bubble columns. A large body of literature is available to predict the bubble velocity and/or shape from empirical and semiempirical studies. Several reviews or books on the subject are available. The most important ones are Clift et al. (1978), Perry and Green (1984), Grace and Waigeri (1986), and Happel and Brenner (1991).

Nevertheless, the hydrodynamics of the gas–liquid flow are still not totally understood. It was shown in the past that due to the deformability of the particle, different shapes could be observed (spherical, ellipsoid, spherical cap) resulting in different flow regimes and a variety of correlations describing these experimental observations. This leads to important complications and discrepancies when evaluating, say, the terminal velocity or the drag coefficient for a specific gas–liquid system.

The objective of this article is to propose a simple correlation for the motion of a single gas bubble in a "pure" and viscous Newtonian fluid of infinite extent. The outline of the article is as follows. After a review of the most important previous works on the subject of bubble motion correlations, a presentation of the data bank on bubble velocities (U) as a function of their spherical volume equivalent diameters (d) is made which includes experimental results from the last century. The bases of the proposed model are then exposed and the results are compared with past experiments and correlations. A discussion is also given about the limiting cases reproduced by the model.

Previous Work

In the actual literature, several corralations have been developed to predict the motion of a single gas bubble in a viscous fluid. This section presents in chronological order the most important correlations available until recently. In all cases the assumption is made that a gas bubble is equivalent to a drop of negligible density (ρ) and viscosity (μ) with respect to the suspending fluid. It is also supposed that the liquids are free of impurities, in which cases the bubbles are known to behave as rigid sphere and to follow Stokes' law at small Reynolds numbers (Stokes, 1851).

In 1911, two fundamental articles were published relating the drag coefficient (C_d) to the Reynolds number (Re). This famous relation developed simultaneously by Hadamard (1911) and by Rybcszynski (1911) was obtained for Reynolds numbers much less than one (laminar flow). A few years later, Levich in 1949 (see also Levich, 1962) recalculated the drag coefficient, assuming that the Reynolds number was high (turbulent flow). The relation obtained relates once again the drag coefficient to the Reynolds number, but with a higher coefficient (due to the viscous dissipation). The following year, Davies and Taylor (1950) obtained a relation between the velocity and the radius of curvature (R_c) for bubbles having a spherical-cap shape. By assuming that the pressure over the front of the spherical-cap bubble is the same as that on a complete spherical bubble, they showed that the terminal velocity was directly proportional to the square root of the radius of curvature (R_c) or the sphere equivalent radius (R) (see Haberman and Morton, 1956). From an extensive experimental study on bubble motion in various liquids, Peebles and Garber (1953) represented their measurements between the drag coefficient and the Reynolds number by breaking down their curves into four distinct regions. Using the irrotational solution approximation, Moore (1959) extended the calculations to the case of a nonspherical bubble using χ (the ratio of the transverse and longitudinal axes of the bubble) as the parameter. The author found that for liquid having Morton numbers (M) less than 10^{-7}, the drag coefficient was in good agreement with their calculations.

After a short period, first-order corrections were applied to existing equations. Moore (1963) reworked the equations of Levich to include a correction as a function of the Reynolds number. Using the boundary layer governing equations on a spherical bubble in a liquid of low viscosity, a corrected equation was developed. The following year, Taylor and Acrivos (1964) reworked the laminar case and introduced to the first order the effect of inertia and deformation via the Reynolds and Weber (We) numbers, respectively. Using a singular perturbation solution, the relation predicts an increase of the drag coefficient as a function of both parameters, with a small predominance of inertia. In 1965, Moore reworked the 1959 calculations to introduce the influence of the bubble deformation through the axis ratio χ. Two years later, Mendelson (1967) discussed that a transition exists for bubble motion in low-viscosity liquids, which represents a change from a viscous to an inviscid flow. By a subdivision of the inviscid regime into a surface-tension-driven flow (Weber number) and a buoyancy-driven flow (Froude number), he proposed a new equation based on the wave analogy. This equation predicted the possible maximum observed for several gas–liquid systems in the graphical representation of the velocity as a function of the radius. The relation can also be seen as a modification of the Davies and Taylor equation to include the effect of surface tension.

Recognizing that a standard C_d vs. Re curve is not practical to use, since the velocity (U) and the diameter (d) are on both axes, Wallis (1974) introduced some dimensionless velocity (V^*) and radius (R^*). Using experimental data from the literature, he observed six distinct regions of different slopes. His set of equations is found to be difficult to use since the transitions between each section of the curve are not well defined. Miyahara and Takahashi (1985), using a limited amount of experimental data taken from the literature, also observed a different behavior for liquids having Morton number values greater or less than 10^{-7}. They used the deformation parameter developed by Tadaki and Maeda (1961), where the ratio of the bubble spherical equivalent diameter (d) over its major axis (A) was related to the Tadaki number (Ta). With these definitions, Miyahara and Takahashi (1985) found that for a modified Reynolds number (Re') greater than 10, the modified drag coefficient (C_d') lies between two given limiting curves.

More recently, Jamialahmadi et al. (1994) proposed a relation based on a root average velocity between the Hadamard-Rybczynski relation and the Mendelson relation. The same year, Karamanev (1994) proposed a relation including explicitly the shape of the bubble via the Tadaki number. The final equation was developed when no internal circulation is present. In 1996, Maxworthy et al. (1996) performed extensive experimental measurements with water/glycerin mixtures between 0 and 100 wt. %. This resulted in a very broad range of physical properties, especially the Morton number ($7.71\times10^{-12} < M < 78.3$). Using their data, they proposed a correlation in two sections based on the value of the Eötvös (Eo) number. Finally, Wesselingh and Bollen (1999) proposed two dimensionless correlations for the velocity of gas bubbles based on reference values for each parameter (dimensionless form). Unforntunately, no information is given on how to select between their "small bubbles"and their "large bubbles" curves, or how to evaluate this transition, the terms "small" or "large" bubble not being clearly quantified.

From all the available literature on the subject of bubble motion, it is very easy to see that no true universal correlation exists to predict the terminal velocity. At best, each equation is valid only in a very limited range. On the other hand, some equations are broken down into several sections to give a better representation of the experimental data. In these cases, the transition between each section is not always clear and can sometimes be tricky to use for design purposes.

In the next section, some experimental data are gathered in order to build a data bank on the motion of gas bubbles in viscous liquids.

Data Bank

To get useful information about the motion of a single gas bubble in a pure and viscous liquid considered quiescent and unbounded, the literature was searched to get the broadest range of physical properties (viscosity, density, and surface tension) as possible. Two main difficulties arise while analyzing the data, namely the effect of impurities and the proximity of the solid walls of the liquid container.

The influence of impurities is very hard to isolate, since liquids can always be contaminated, particularly for earlier studies. This is especially true in the case of water, which however pure we may have it, a small amount of surface-active contaminant can be present. For this reason most of the studies using water as the liquid phase were not taken into account for the correlation. In most cases, no information is given about the "quality" of the water or only tap water was used. Only the data of Duineveld (1995) were kept, for he used "hyperclean" water produced with a Millipore purification system.

The influence of walls was not a parameter in this study. To minimize its effect, only data that had a diameter ratio (λ) less than 10% were used. That is to say when the value of $d/D < 0.1$, where D is the diameter of the container for cylindrical columns or the length of one side for square ducts. Nevertheless, a small correction factor (< 2%) was applied to all remaining data following the simple correlation given in Clift et al. (1978).

$$U_\infty = U/[1-\lambda^2]^{3/2}. \tag{1}$$

In total, the data bank consists of 897 points taken from 19 studies spanning a period from 1900 to 1996. A very large range of physical properties and hydrodynamic regimes is covered:

$$722 < \rho < 1{,}380 \text{ kg/m}^3$$

$$2.2\times 10^{-4} < \mu < 18 \text{ Pa}\cdot\text{s}$$

$$15.9 < \sigma < 91.0 \text{ mN/m}$$

$$1.9\times 10^{-7} < Re < 1.1\times 10^{4}$$

$$1.0\times 10^{-11} < M < 1.0\times 10^{7}.$$

Table 1 summarizes all the studies with the required information (when available) for each set of data. In some cases, all the physical properties are not directly given. Instead, ratios like μ/ρ and σ/ρ can be deduced. This information is nonetheless sufficient when working in a dimensionless form, as will be done in the next sections.

Model

The proposed model is based on dimensional analysis. A general equation for the motion of a gas bubble is generated from the relevant physical parameters. In a mathematical form, a general function G_1 is wanted, which satisfies

$$G_1(d, U, \rho, \sigma, \mu, g) = 0, \tag{2}$$

where g is the gravitational acceleration. In this particular case, the Buckingham Π theorem of dimensional analysis will give three distinct dimensionless numbers. It is more difficult to represent and to extract a correlation from experimental data in three dimensions. Following the works of Abou-El-Hassan (1983, 1986), a way to lump these parameters is required to reduce the number of variables from six to five, giving only two dimensionless numbers. The lumped parameters were given as

- The bubble equivalent diameter (d)
- The kinematic viscosity of the liquid phase (μ/ρ)
- A physical parameter obtained from the Mendelson wave analogy ($\sigma\mu^2$)
- The buoyancy force per unit volume (ρg)
- The momentum per unit volume (ρU).

The new function G_2 should satisfy the following equation:

$$G_2(d, \mu/\rho, \sigma\mu^2, \rho g, \rho U) = 0. \tag{3}$$

Using standard dimensional analysis, the two new numbers obtained are the flow number (F) and the velocity number (V), defined as

$$F = g\left[\frac{d^8\rho^5}{\sigma\mu^4}\right]^{1/3} = Eo\left(\frac{Re}{Ca}\right)^{2/3} \tag{4}$$

$$V = U\left[\frac{d^2\rho^2}{\sigma\mu}\right]^{1/3} = (Re^2\,Ca)^{1/3}, \tag{5}$$

where Ca, Eo, and Re are the capillary, Eötvös, and Reynolds numbers, respectively, and defined as

$$Ca = \frac{\mu U}{\sigma} \tag{6}$$

$$Eo = \frac{\rho g d^2}{\sigma} \tag{7}$$

$$Re = \frac{\rho U d}{\mu}. \tag{8}$$

A similar analysis was performed by Wanchoo et al. (1997), which correlated nicely their experimental data for single vapor bubbles collapsing in another immiscible liquid.

In his original article, Abou-El-Hassan proposed a relation in the form of

$$V = 0.75\,[\log(F)]^2 \tag{9}$$

for a limited range of physical parameters, especially for the viscosity (30 times less). As seen on Figure 1, the application of this relation to a larger range of experimental conditions reveals some discrepancies at low and high values of the flow number. Equation 9 underpredicts the value of the velocity number V at high F, while the relation breaks down when $F < 1$ (negative log values). Based on Figure 1, a new correlation is proposed in the form of

$$V = \frac{aF^b}{1 + cF^d}. \tag{10}$$

Using a nonlinear fitting package (Sigmaplot 5.0, SPSS Scientific), the value of each parameter a, b, c and d were found to be

$$a = \frac{1}{12}, \qquad b = 1, \qquad c = \frac{49}{1{,}000}, \qquad d = \frac{3}{4}. \tag{11}$$

Figure 1 shows the proposed correlation obtained by introducing Eq. 11 into Eq. 10. A good agreement is seen for the whole range of the flow parameter available.

For the sake of comparison, a second correlation was tried to compare with Eq. 10, as

$$V = \frac{aF^b}{[1 + cF]^d}. \tag{12}$$

In this case, the parameters were found to be

$$a = \frac{1}{12}, \qquad b = 1, \qquad c = \frac{37}{2{,}000}, \qquad d = \frac{3}{4}. \tag{13}$$

Using statistical analysis, the results obtained from Eqs. 12–13 could not be differentiated from the ones obtained by Eqs. 10–11. For the sake of simplicity, only Eqs. 10–11 are discussed next, but the conclusions are found to be equally applicable to both correlations.

Discussion

A great advantage of Eq. 10 over curves like C_d vs. Re is its simplicity in computing explicitly the velocity from the diameter, the viscosity, the density, and the surface tension: U is only present in the velocity number (V). Figure 2 compares the predicted velocity as a function of the measured velocity, which covers six decades. Besides, evident scatter attributed to experimental measurements, most of the data points fall

Table 1. Studies Included in the Experimental Data Bank

Reference	Liquid	T (°C)	D (cm)	ρ (kg/m^3)	σ (mN/m)	μ (mPa·s)
Allen (1900)	Aniline	10	3.5	1,038	44.1	6.02
Angelino (1966)	Dipropylene/Tripropylene glycol	18	18.5	1,023	34.2	105
Arnold (1911)	Olive oil	22	?*	925	34.7	73
Bond and Newton (1928)	Golden Syrup	17	4.75	1,480	91	18,000
Bryn (1933)	58/42 H_2O/Glycerin	18	15	1,169	71.1	4.56
Davies and Taylor (1950)	Nitrobenzene	14	61	1,200	43.9	1.8
Duineveld (1995)	Purified water	19.6	50	998	72.8	1.01
Garner and Hammerton (1954)	Hydrocarbon white oil	?**	2	n/a†	n/a†	n/a†
Gailhbaud and Zortea (1969)	Saccharose aceto-isobutyrate (SAIB)	50††	45	1,146	40	16,600
Gorring and Katz (1962)	29.4/70.6 H_2O/glycerin	24	9.4	n/a‡	n/a‡	n/a‡
	n-Heptane	24	9.4	n/a‡‡	n/a‡‡	n/a‡‡
Kojima et al. (1968)	Castor oil	31.8	20	953	38.8	376
	Glycerin	15.9	20	1,270	63.6	1,960
	Glycerin	20	20	1,260	63.4	1,330
	Glycerin	35	20	1,250	62.6	364
	Glycerin	25.3	20	1,260	63.2	378
	Glycerin	25.4	20	1,260	63.2	157
	Corn syrup	13.5	20	1,380	95.6	10,380
	Corn syrup	20.5	20	1,380	86.0	4,580
	Corn syrup	29.2	20	1,380	81.0	1,940
Kubota et al. (1967)	47/53 H_2O/glycerin	20	8	1,135	70.3	6.86
	34/66 H_2O/glycerin	20	8	1,170	68.1	15.4
	20.5/79.5 H_2O/glycerin	20	8	1,206	65.9	52.9
	Ethyl acetate	20	8	901	23.8	0.449
	Acetic acid	20	8	1,049	27.3	1.22
	Ethanol	14	8	789	23.4	1.30
Maxworthy et al. (1996)	0.5/99.5 H_2O/glycerin	20.5	10.2	1,260	62.4	1,250
	0.2/99.8 H_2O/glycerin	30	10.2	1,252	62.1	580
	0.2/99.8 H_2O/glycerin	40	10.2	1,247	61.5	275
	20/80 H_2O/glycerin	20	10.2	1,209	65.5	60.1
	40/60 H_2O/glycerin	20	10.2	1,154	67.8	9.45
	60/40 H_2O/glycerin	20	10.2	1,099	69.6	3.84
	70/30 H_2O/glycerin	20	10.2	1,073	70.3	2.55
	80/20 H_2O/glycerin	20	10.2	1,047	70.9	1.78
	90/10 H_2O/glycerin	20	10.2	1,022	71.3	1.32
Pan and Acrivos (1968)	Zelorene	29	10.2	910	26.5	737
Peebles and Garber (1953)	Ethyl ether	?**	2.6	722	15.9	0.23
	5.5/94.5 Ethyl acetate/cottonseed oil	?**	2.6	905	34.1	31
Rodrigue (1996)	8/92 H_2O/glycerin	24.6	8	1,238	66.3	208
	5/95 H_2O/glycerin	24.2	8	1,245	68.7	337
	1/99 H_2O/glycerin	24.8	8	1,254	65.8	737
Tadaki and Maeda (1961)	Isoamyl alcohol	12.5	10	813	24.5	5.6
	25/75 H_2O/glycerin	15.7	10	1,200	58.7	50
	Toluene	11.8	10	912	29.5	0.65
	92/8 H_2O/ethanol	12.8	10	970	52.8	1.8
	Ethanol	12.5	10	796	23.1	1.41
	Nitrobenzene	14.3	10	1,082	44.6	2.34
	20/80 H_2O/acetic acid	11.7	10	1,070	32.1	3.5
	70/30 H_2O/ethanol	12.7	10	960	33.8	3.6
	Ethyl acetate	13	10	909	25.0	0.489
Tsuge and Hibino (1971)	50/50 H_2O/glycerin	23.2	8	1,127	67.1	5.33
	75/25 H_2O/glycerin	21.5	8	1,056	69.4	1.88
	Isobutanol	26.1	8	799	20.2	3.05
	80/20 H_2O/methanol	23.8	8	968	43.5	1.45
	Ethanol	26.9	8	784	22.1	1.00
	Nitrobenzene	27.8	8	1,194	42.7	1.68
Uno and Kintner (1956)	Diethylene glycol	28	15.3	1,111	46.8	24.1

*Not given in the article, but assumed to be very large.
**Not given in the article.
†Not directly given, but the values of μ/ρ and σ/ρ are 1.85×10^{-4} m^2/s and 1.41×10^{-5} m^3/s^2, respectively.
††Tests performed at 50°C, but properties only given at 25°C except the viscosity.
‡Not directly given, but the values of μ/ρ and σ/ρ are 1.66×10^{-5} m^2/s and 6.60×10^{-5} m^3/s^2, respectively.
‡‡Not directly given, but the values of μ/ρ and σ/ρ are 5.70×10^{-7} m^2/s and 1.72×10^{-5} m^3/s^2, respectively.

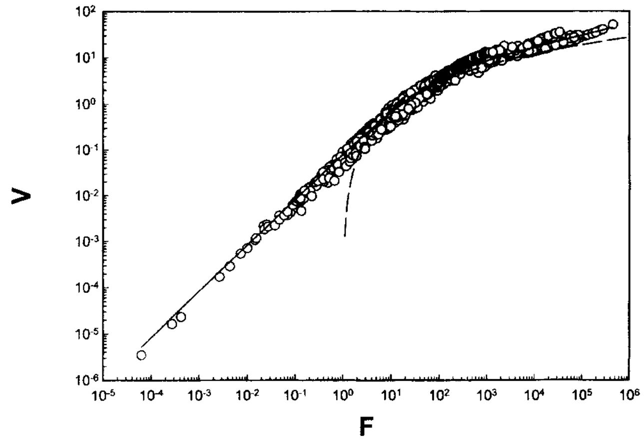

Figure 1. Velocity number (V) as a function of the flow number (F).

○: Experimental data from Table 1; ---: Eq. 9; ——: Eqs. 10–11.

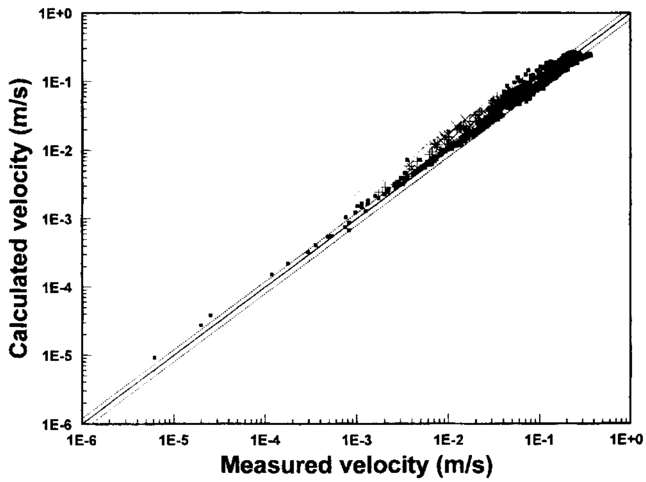

Figure 2. Measured vs. calculated velocity from Eqs. 10 11.

×: Data for SAIB from Gailhbaud and Zortea (1969); +: data for aniline from Allen (1900); ■: remaining data from Table 1. The dotted lines represent ± 20% deviation.

between 20% of the prediction, with the exception of two sets: the data for aniline by Allen (1900) and some of the data for SAIB by Gailhbaud and Zortea (1969). In the first case, all the measured velocities fall around 50% below the predicted ones. This points to a possible contamination of the fluid (immobilized interface reproducing rigid sphere behavior). In the second case, the discrepancies between the model and the data can be attributed to the physical properties, which were not given in the original article at the same temperature as of the performed experiments (25 vs. 50°C). Nonetheless, the correlation is very acceptable for the whole range of conditions covered.

It can be shown that Eq. 10 can reproduce classic limiting cases of bubble hydrodynamics. In a general way, it is possible to relate the velocity number to the flow number in a power law manner as

$$V \propto F^n, \tag{14}$$

where n is a function of the flow regime. In this way, Eq. 10 agrees with earlier studies on bubble velocity. It can be shown that three flow regimes are included as limiting cases in the new model.

The first regime is for $F \ll 1$, where Eq. 10 simplifies to

$$V = \frac{1}{12}F \quad \text{or} \quad U = \frac{1}{12}\frac{\rho g d^2}{\mu}. \tag{15}$$

This corresponds to $n = 1$, the flow is laminar, and the model reproduces exactly the Hadamard-Rybczynski correlation. On the other hand, when $F \gg 1$, Eq. 10 simplifies to

$$V \propto F^{1/4} \quad \text{or} \quad U \propto \mu^0 d^0. \tag{16}$$

This corresponds to $n = 1/4$, the flow is turbulent, and the model predicts a velocity that is independent of the viscosity and the diameter. At an intermediate value of n (between 1 and 1/4), a value of 7/16 will give:

$$U \propto g^{7/16} d^{1/2}, \tag{17}$$

which is very close to the prediction of Davies and Taylor (1950) and represents a bubble having a spherical-cap shape. The fact that these three limiting cases are included in the proposed model gives some weight to its validity.

Conclusions

The main contribution of this work is the development of a generalized correlation for bubble velocities. From a single equation, it is possible to calculate the velocity of a gas bubble rising steadily in a quiescent viscous Newtonian and unbounded liquid free of contamination. These velocities are in good agreement with experimental measurements (within 20%) for a very broad range of density, surface tension, and viscosity of the liquid phase. The model covers 6 decades in velocities, 11 decades in Reynolds numbers, and 18 decades in Morton numbers. This is the largest range of these parameters encompassed in the literature so far.

The model is also in agreement with earlier well-known correlations. Its ability to predict three distinct hydrodynamic regimes, namely the laminar regime ($U \propto \mu^{-1} d^2$), the turbulent regime ($U \propto \mu^0 d^0$), and an intermediate regime ($U \propto g^{7/16} d^{1/2}$) brings some additional support. This enables the model to be highly useful in design purposes due to its simplicity and precision over such a large range of conditions.

Acknowledgments

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